L I P 6 C N R S

Automatic Theorem Proving

Resolution in Propositional Logic
Unification
Resolution in First Order Logic

ACASA – Faculté des sciences de Sorbonne Université

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Resolution in Propositional Logic

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Resolution in First Order Logic

Resolution Rule in Propositional Logic

- Been given two clauses C_1 and C_2
- Been given an atomic proposition such that $A \in C_1$ and $\neg A \in C_2$
- The resolution of C_1 and C_2 by A and $\neg A$ is:
  $C = res(C_1, C_2; A, \neg A)$
  $= [C_1 - \{A\}] \vee [C_2 - \{\neg A\}]$

Example: if $C_1 = \neg man \vee mortal$,
$C_2 = \neg socrate \vee man$, $C_3 =$ socrate and
$C_4 = \neg mortal$

$C = res(C_3, C_2; socrate, \neg socrate) = man$

$C' = res(C_1, C_4 ; mortal, \neg mortal) = \neg man$

Resolution (*example*)

$C_1 = \neg \text{man} \vee \text{mortal} = \text{man} \supset \text{mortal}$

$C_2 = \neg \text{socrate} \vee \text{man} = \text{socrate} \supset \text{man}$

$C_3 = \text{socrate}$, $C_4 = \neg \text{mortal}$,

$C = \text{res}(C_3, C_2; \text{socrate}, \neg \text{socrate}) = \text{man}$

$C' = \text{res}(C_1, C_4 ; \text{mortal}, \neg \text{mortal}) = \neg \text{man}$

- The resolution is more general than the *Modus Ponens* (A, A ⊃ B/B) and the *Modus Tolens* (¬B, A ⊃ B/¬A)

Generality of the Resolution

- To prove $S \models C$, it is sufficient to prove that $S \cup \{\neg C\}$ is contradictory, i.e. that $S \cup \{\neg C\} \models \square$

$\square$ denotes the empty clause, i.e. the false

- **Theorem**: $S \models C$ if and only if it is possible to derive the empty clause by iterative application of the resolution rule on $S \cup \{\neg C\}$

Example

- Consider S = {C_1, C_2, C_3}

C_1 = ¬man ∨ mortal = man ⊃ mortal

C_2= ¬socrate ∨ man = socrate ⊃ man

C_3= socrate, C_4= ¬mortal,

- To prove S ⊨ mortel, it is sufficient to derive the empty clause, i.e. □, from S ∪{C_4}

C_5: ¬man res(C_1, C_4; mortal, ¬mortal)

C_6: ¬socrate res(C_2, C_5; man, ¬man)

C_5: □ res(C_3, C_6; socrate, ¬socrate)

QED

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Substitution

Definition:

- Being $\mathcal{V}$ the set of variables.
- Being $\mathcal{F}$ the set of functions.
- Being $\mathcal{T}$ the set of terms built on $\mathcal{V}$ and $\mathcal{F}$.

A *substitution* σ is an application from $\mathcal{V}$ to the set of terms $\mathcal{T}$ which is the identity *almost everywhere*

A *substitution* is characterized by a finite set of pairs "x_i / t_i", x_i being a variable and t_i a term.

It is denoted $\sigma = \{v_1/t_1, \ldots, v_n/t_n\}$

Example: $\sigma = \{x/3, y/(u - 7)\}$

Term instance

Definition:

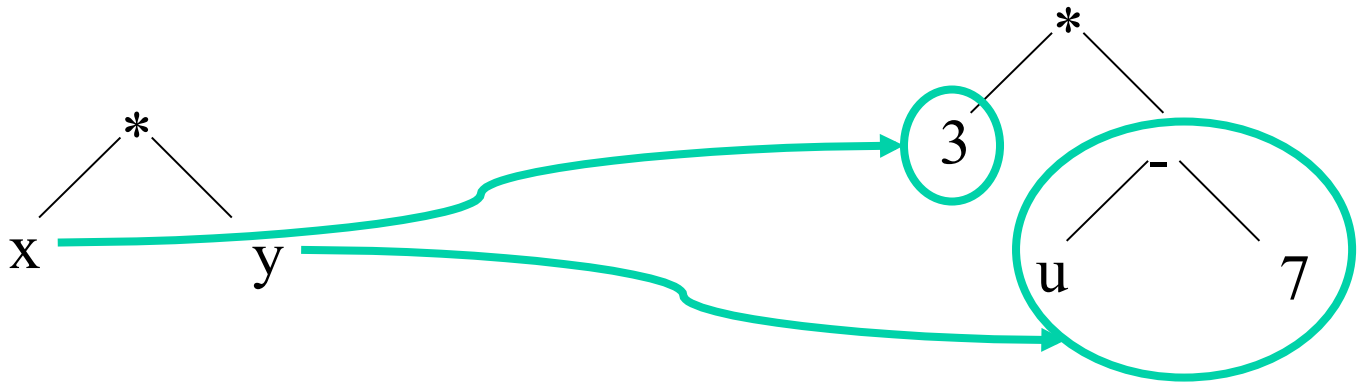
- Being E a term of $\mathcal{T}$ (set of terms built on $\mathcal{V}$ and $\mathcal{F}$)
- Being a substitution σ of variables $\mathcal{V}$ by terms of $\mathcal{T}$
  $\sigma = \{v_1/t_1, \ldots, v_n/t_n\}$

Being $E\sigma$ the expression built by replacing each occurrence of free variables v_i of E by t_i.

$E\sigma$ is called an *instance* of E

Term instance (following)

Example: if σ = {x/3, y/(u – 7)}et E = (x*y)
then Eσ = (3 * (u – 7)) is an instance of E



Substitution composition

- Being $\theta=\{t_1/x_1,\ldots, t_n/x_n\}$ and $\lambda=\{u_1/y_1,\ldots, u_m/y_m\}$ two substitutions.

- The *composition* of θ and λ is the substitution $\lambda\circ\theta$ which is obtained from the set $\{t_1\lambda/x_1,\ldots, t_n\lambda/x_n, u_1/y_1,\ldots, u_m/y_m\}$ by deleting all the $t_j\lambda/x_j$ such that $t_j\lambda=x_j$ and all the u_i/y_i such that $y_i\in\{x_1, \ldots, x_n\}$

Substitution composition

- The *composition* of two substitution θ and λ is the substitution $\lambda\circ\theta$ which is obtained from $\{t_1\lambda/x_1,\ldots, t_n\lambda/x_n, u_1/y_1,\ldots, u_m/y_m\}$ by deleting all the $t_j\lambda/x_j$ such that $t_j\lambda=x_j$ and all the u_i/y_i such that $y_i\in\{x_1,\ldots, x_n\}$

Example: $\theta=\{t_1/x_1, t_2/x_n\}=\{f(y)/x, z/y\}$ et
$\lambda=\{u_1/y_1, u_2/y_2, u_3/y_3\} = \{a/x, b/y, y/z\}$

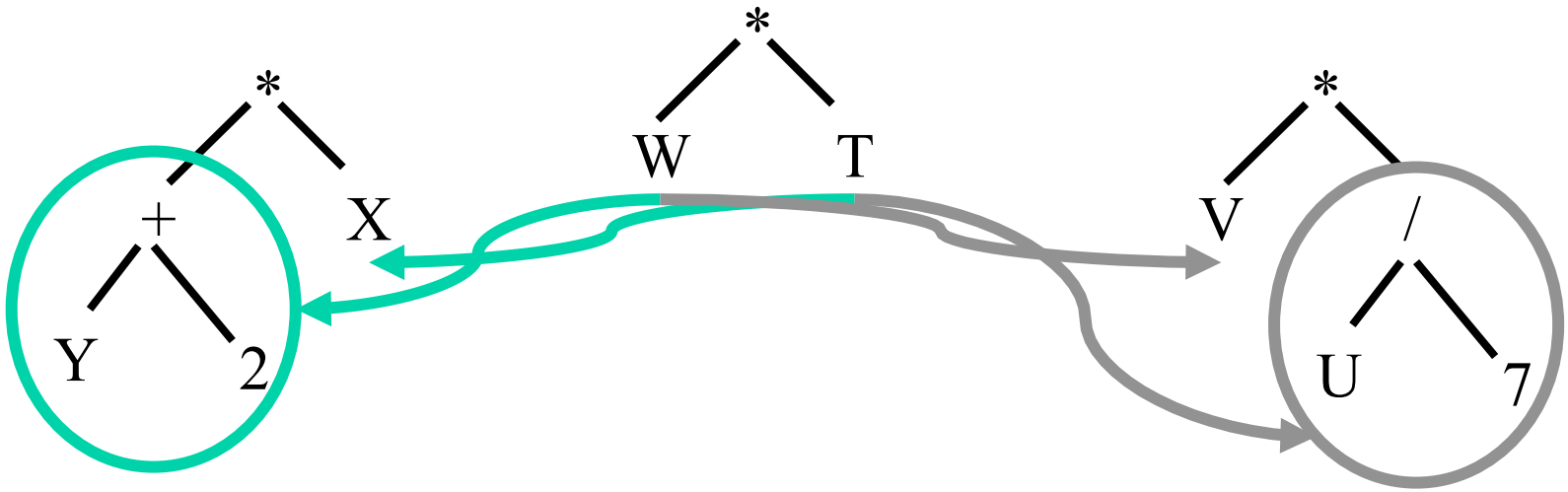
$\{t_1\lambda/x_1,\ldots, t_n\lambda/x_n, u_1/y_1,\ldots, u_m/y_m\} =$
$\{f(b)/x, \underline{y/y}, \underline{a/x}, \underline{b/y}, y/z\} = \{f(b)/x, y/z\}$

$\theta\circ\lambda = \{f(b)/x, y/z\}$

Pattern matching

Is a term an instance of another?

Pattern matching: the term t_1 match with the term t_2 if and only if there exists a substitution σ such that: $\mathbf{t_1\sigma = t_2}$



Pattern matching algorithm

filtering(t1, t2)

If t1 = t2 **then** σ := {} **return;**

If t1 = () or t2 = () **then** σ := fail; **return;**

If var(t1) **then** σ := {{t1 / t2}}; **return;**

If atom(t1) or atom(t2) **then** σ := fail; **return;**

σ_1 := filtering(head (t1) , head(t2));

If σ_1 = fail **then** σ := fail ; **return;**

else σ_2 := filtering(substitute(σ_1 , tail(t1)) , tail(t2));

If σ_2 = fail **then** σ := fail; **return;**

else σ := composition (σ_1 , σ_2)

Representing a term with a list

- **S-expression**:
  - An *S-expression* is either an *atom (character string)*
  - Or a *sequence* (possibly empty) of S-expression
- *Examples*:
  - []
  - ‘a’, ‘Salut’, ‘Insoluble’, ‘Rien!’, 421
  - [‘a’, [‘Salut’, ‘à’, ‘toi’], 421, [‘Rien’, [‘vraiment rien!’]]]
- **Term**:
  - **Definition**: set of functions $\mathcal{F}$, set of variables $\mathcal{V}$
  - **Representation with a S-expression**: (3 * (?x + (?y - ?x)))
    [‘*’, 3, [‘+’, ‘?x’, [‘-’, ‘?y’, ‘?x’]]]

```
def filtrage_(t1, t2, v1):
    if t1 == t2:
      return { }
  if var(t1):
      if t1 in v1:
         sigma = {}
         sigma[t1] = t2
      else: sigma = 'echec'
      return sigma
  if ( t1 == [] or t2 == []):
      return 'echec'
  if ( type(t1) <> type([]) or  type(t2) <> type([])):
      return 'echec'
  sigma1 = filtrage_(t1[0], t2[0], v1)
  if (sigma1 <> 'echec' and len(t1) > 1 and  len(t2) > 1 ):
      sigma2 = filtrage_(appliquer_substitution(t1[1:], sigma1), t2[1:], v1)
      return composer_substitution(sigma2, sigma1)
  elif t1[1:] <> t2[1:]:
      return 'echec'
  else: return sigma1
```

)

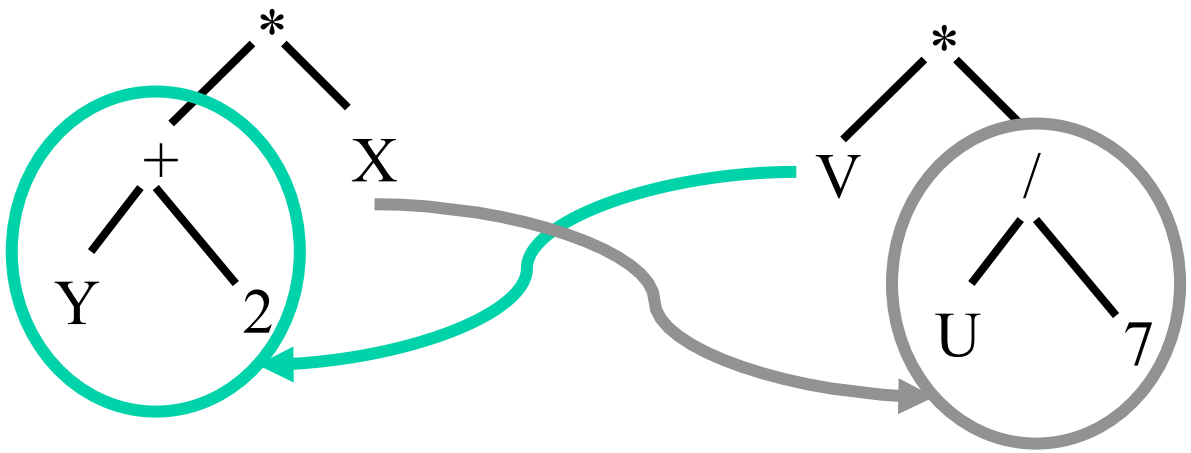
2, vars1)

Unification

Unification: the terms t_1 and t_2 unify if and only if there exists a substitution σ such that:

$t_1\sigma = t_2\sigma$

Pattern matching ⇒ Unification



Most General "Unifier"

Definition: a substitution σ is an *unifier* of the set of terms $\{t_1, t_2, \ldots t_n\}$ if and only if

$t_1 \sigma = t_2 \sigma = \ldots = t_n \sigma$

Definition: a unifier σ of a set of terms $\{t_1, t_2, \ldots t_n\}$ is *the most general unifier* (*mgu*) if and only if for all unifier θ of this set, there exists λ such that $\theta = \sigma \circ \lambda$

Most General "Unifier"

Example: S = {P(g(z), y), P(x, f(u))}

S is unifiable because the substitution
θ = {x/g(a), y/f(b), z/a, u/b} is a unifier S

θ is not the most general unifier because there exists a unifier ζ = {x/g(z), y/f(b), u/b} such that θ = ζ ° {z/a}

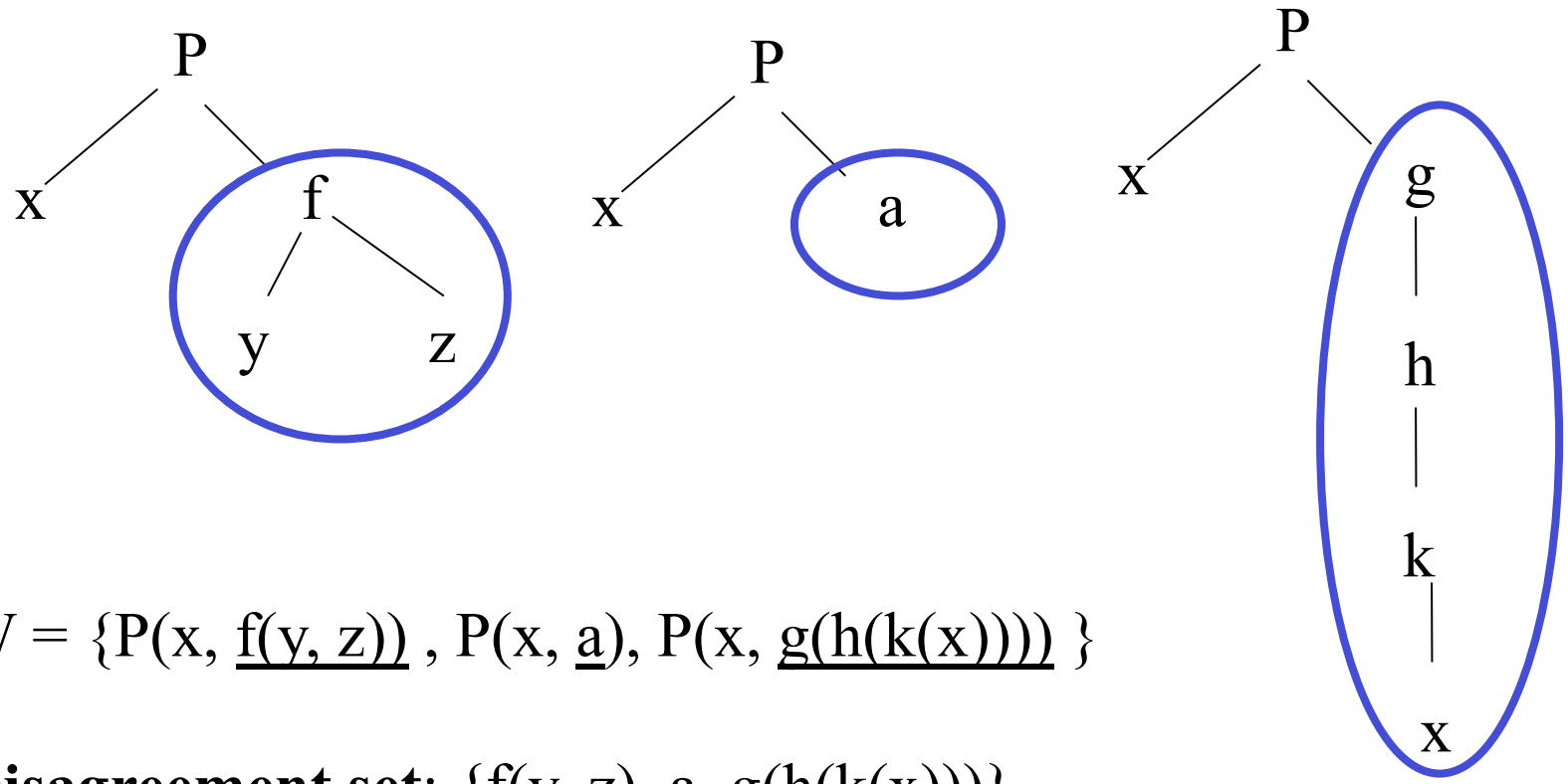
The unifier σ = {x/g(z), y/f(u)} is the *most general unifier* of this set

Unification Algorithm (*preliminaries*)

Definition: *a disagreement set* of a set of terms W is obtained by detecting the first difference in non identical terms (depth first search).

Example: W = {P(x, f(y, z)) , P(x, a), P(x, g(h(k(x)))) }

Disagreement set: *example*



W = {P(x, f(y, z)) , P(x, a), P(x, g(h(k(x)))) }

Disagreement set: {f(y, z), a, g(h(k(x)))}

Unification algorithm for a set of terms W

Step 1: $k := 0$, $W_k := W$, $\sigma_k := \varepsilon$

Step 2: if W_k is a singleton then stop; σ_k is the MGU (*most general unifier)* of W.
else, find the disagreement set D_k of W_k

Step 3: if there exists a variable v_k and a term t_k belonging to D_k such that v_k does not appear in t_k, then go to step 4.
else, stop; W is not unifiable

Step 4: $\sigma_{k+1} := \sigma_k \{v_k / t_k\}$; $W_{k+1} := W_k \{v_k / t_k\}$ *[note that* $W_{k+1} = W\sigma_{k+1}$ *]*

Step 5: $k := k+1$; go to step 2

Unification algorithm: *example*

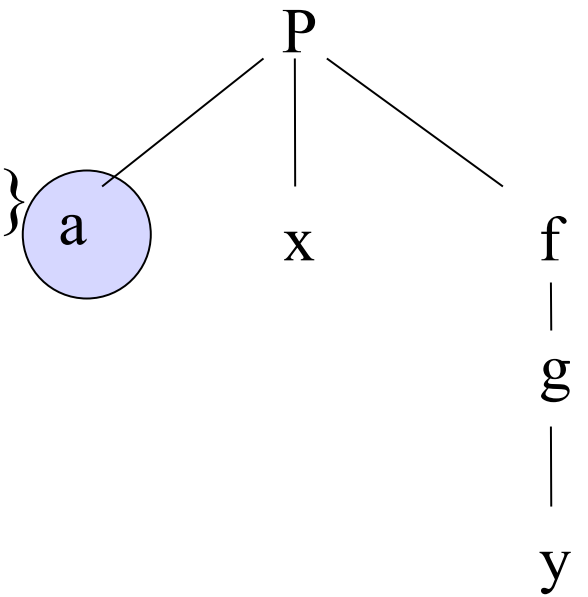
$W = \{ P(a, x, f(g(y))), P(z, f(z), f(u))\}$

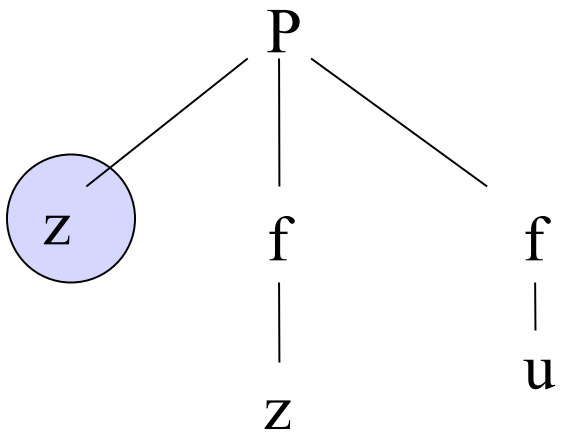
$k = 0$

$W_k = W,$

$\sigma_k = \varepsilon$

$D_k = \{a, z\}$





Unification algorithm: *example*

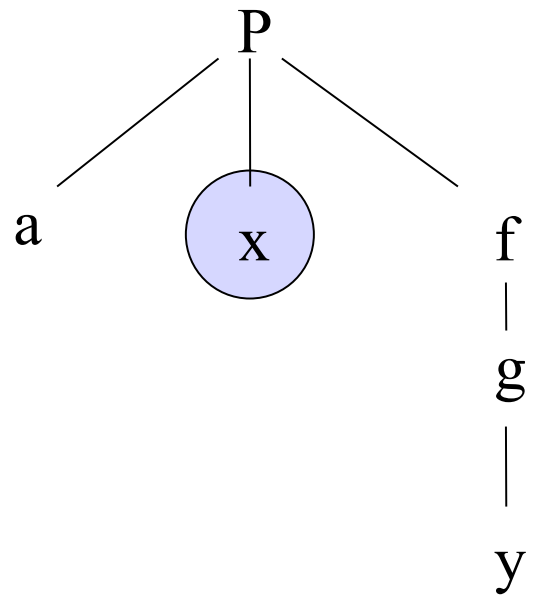
$W = \{ P(a, x, f(g(y))), P(z, f(z), f(u))\}$

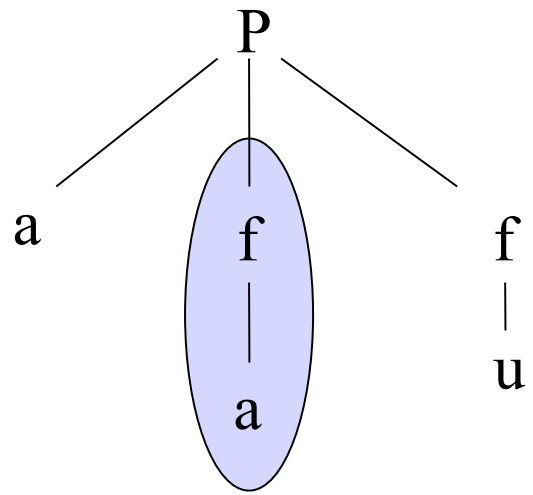
$k = 1$

$W_k = \{ P(a, x, f(g(y))), P(a, f(a), f(u))\}$

$\sigma_k = \{z / a\}$

$D_k = \{x, f(a)\}$





Unification algorithm: *example*

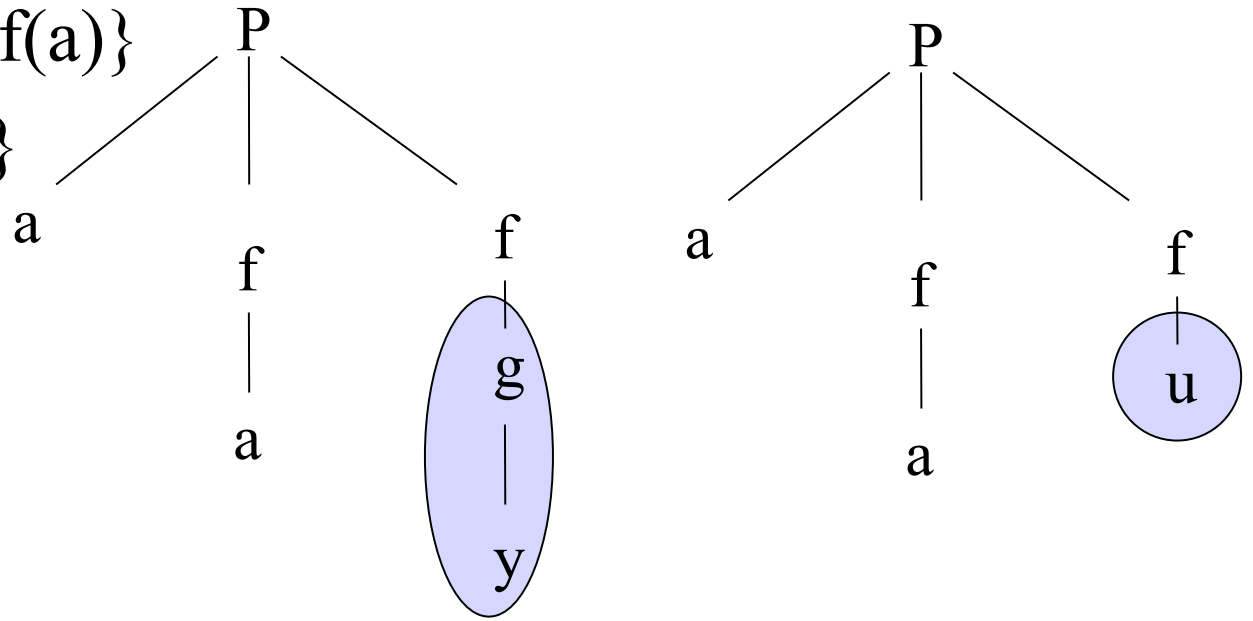
$W = \{ P(a, x, f(g(y))), P(z, f(z), f(u))\}$

$k = 2$

$W_k = \{ P(a, f(a), f(g(y))), P(a, f(a), f(u))\}$

$\sigma_k = \{z / a, x/f(a)\}$

$D_k = \{g(y), u \}$



Unification algorithm: *example*

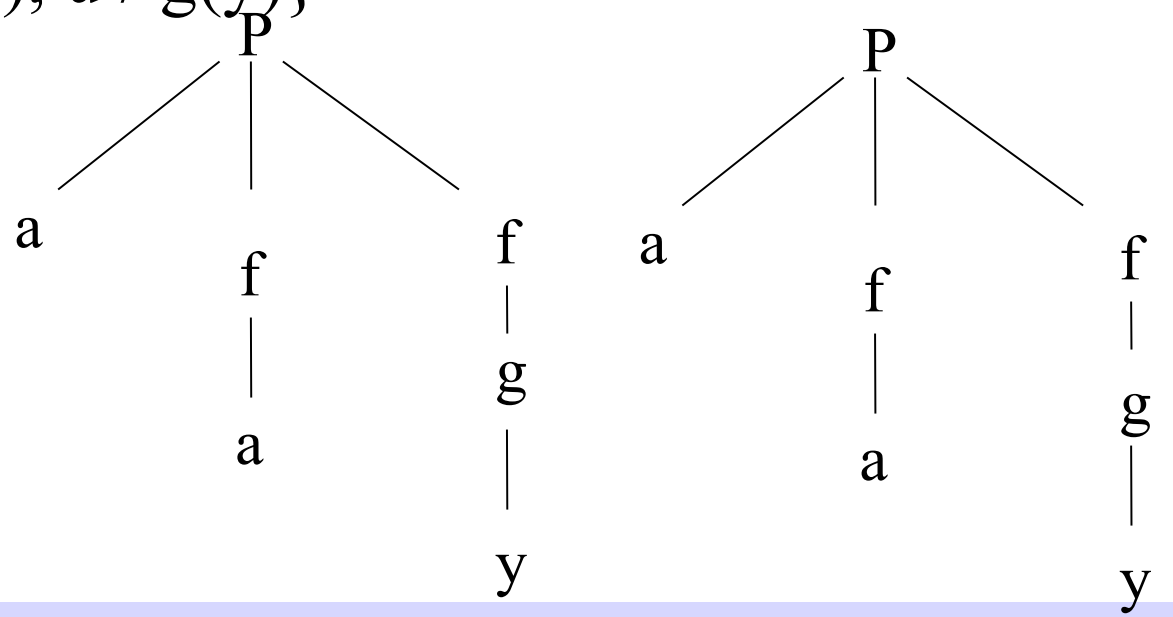
$W = \{ P(a, x, f(g(y))), P(z, f(z), f(u))\}$

$k = 3$

$W_k = \{ P(a, f(a), f(g(y))) \}$

$\sigma_k = \{z / a, x/f(a), u / g(y)\}$

$D_k = \{\}$



Unification Theorem

Theorem: if W is a non empty set of unifiable terms, then the algorithm always finishes at step 2 and the last substitution σ_k is the most general unifier of W.

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General Resolution

Definition:

- Being given two clauses A and B of which variables are disjoint (if needed, rename variable),
- Being given a literal L_1 of A and a literal L_2 of B such that either L_1 and $\neg L_2$ or $\neg L_1$ and L_2 unify with the substitution σ.

C is said to be the resolvant of A and B by L_1 and L_2, which is noted:

$$\mathbf{C = Res(A, B; L_1, L_2) = (A\sigma - \{L_1\sigma\}) \cup (B\sigma - \{L_2\sigma\})}$$

Examples:

Res($\underline{R(x, y)}$, $\underline{\neg R(u, v)}$; R(x, y), ¬R(u, v)) =∅

Res($\underline{P(x)} \vee P(y)$, $\underline{\neg P(z)} \vee \neg P(r)$; P(x), ¬P(z)) = $P(y) \vee \neg P(r)$

Res($\underline{man(Socrates)}$, $\underline{\neg man(z)} \vee mortal(z)$;
man(Socrates), ¬man(z) = mortal(Socrates)

Resolution : example

C_1: vénéneux(X) ∨ ¬champignon(X) ∨ ¬non-comestible(X)

C_2: toxique(U) ∨ ¬vénéneux(U)

C_3: ¬toxique(a)

Rés(C_1, C_2 ; vénéneux(X) , ¬vénéneux(U)) =

Rés(**vénéneux(X)** ∨ ¬champignon(X) ∨ ¬non-comestible(X),
toxique(U) ∨ **¬vénéneux(U)**
; vénéneux(X) , ¬vénéneux(U)) =

¬champignon(X) ∨ ¬non-comestible(X) ∨ toxique(X)

C_1 means **champignon(X) ∧ non-comestible(X) ⇒ vénéneux(X)**

C_2 means **vénéneux(U) ⇒ toxique(U)**

La resolvant of C_1 and C_2 means
champignon(X) ∧ non-comestible(X) ⇒ toxique(X)

Resolution : example (following)

C_1: vénéneux(X) ∨ ¬champignon(X) ∨ ¬non-comestible(X)

C_2: toxique(U) ∨ ¬vénéneux(U)

C_3: ¬toxique(a)

Res(C_2, C_3 ; toxique(U) , ¬toxique(a)) =

Res(**toxique(U)** ∨ ¬vénéneux(U),

¬toxique(a) ,

; toxique(U) , ¬toxique(a)) =

¬vénéneux(a),

C_2 means **vénéneux(U) ⇒ toxique(U)**, which is equivalent to **¬toxique(U) ⇒ ¬vénéneux(U)**,

C_3 means **¬toxique(a)** (in other words that a is not toxic)

The resolvant of C_2 and C_3 means **¬vénéneux(a)** in other words that a is not venenous.

Resolution Theorem

Resolution theorem: being S a set of clauses,

S is unsatisfiable if and only if there exists a R-refutation de S,

in other words, if it is possible to derive the empty clause (□ , i.e. false) by iteratively applying the resolution rule, which means that $S \vdash_{Res} \square$.

Automating the Proof

- Why?
  - Proof of programs
  - Artificial intelligence (*mathematical reasoning*)
  - Programming (*logic-based programming language*)
  - Knowledge representation (*artificial intelligence and data bases*)
- How?

How to automatically prove a theorem?

- **Proof by refutation**:

  Being S a set of clauses and C a clause. To prove that $S \vdash C$ it is enough to prove that S and $\neg C$ are unsatisfiable, which is equivalent to refutate S and $\neg C$ i.e. to proof that S, $\neg C \vdash_{Res}$

- **Resolution**:

$$C_k = Res(C_i, C_j; L_{i1}, L_{j2}) = (C_i\sigma - \{L_{i1}\sigma\}) \cup (C_j\sigma - \{L_{j2}\sigma\})$$

- **Choices** C_i, C_j, L_{i1}, L_{j2}
  - Complexity at step k: $(|S| + k)^2 * |C_i| * |C_j|$

Horn clause

Horn clause: *not more* than a positive literal

Positive Horn clause: *exactly* one positive literal

Also called "definite clause"

Negative Horn clause: *no* positive literal

Also called "Query"

Notation: « P :- N_1, N_2, …, N_n. » is a Horn clause.

« P » is the clause **head**

« N_1, N_2, …, N_n. » is the **body** or the **goal** part of the clause

SLD-Resolution

Definition: (SLD: Selection rule, Linear resolution, Definite clauses)

Selection: choice of literals in the clauses

Linear resolution: one of the two clauses comes from the previous application of the resolution

Definite clauses: positive Horn clauses

How to automatically prove a theorem?

- **Proof by refutation**:

  Being S a set of clauses and C a clause. To prove that $S \vdash C$ it is enough to prove that S and $\neg C$ are unsatisfiable, which is equivalent to refutate S and $\neg C$ i.e. to proof that $S, \neg C \vdash_{Res}$

- **Resolution**:

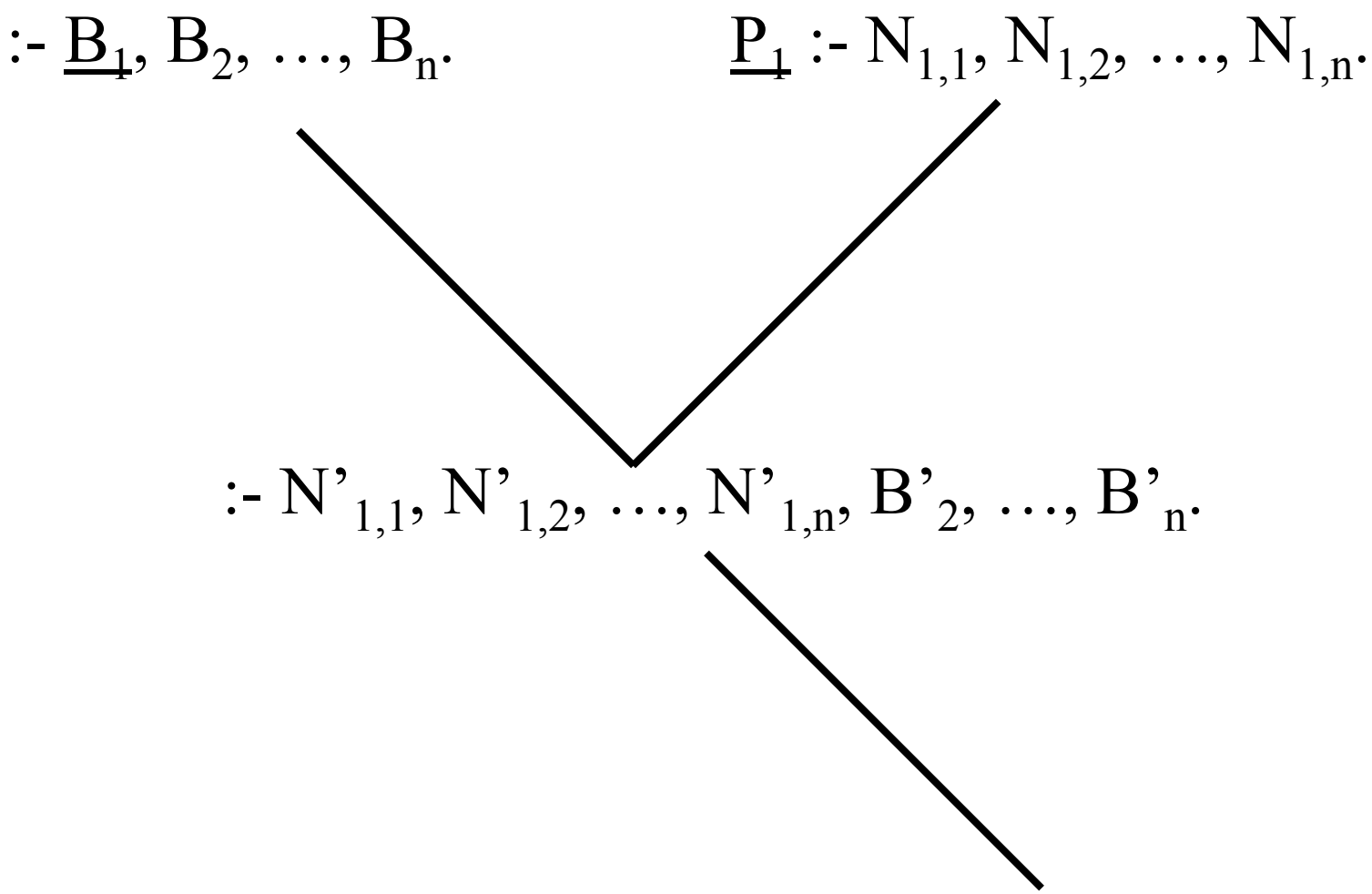
$$C_k = Res(C_i, C_j; L_{i1}, L_{j2}) = (C_i\sigma - \{L_{i1}\sigma\}) \cup (C_j\sigma - \{L_{j2}\sigma\})$$

- **Choices** C_i, C_j, L_{i1}, L_{j2}
  - **Complexity at step k**: $(|S| + k)^2 * |C_i| * |C_j|$

Completeness of SLD resolution

SLD resolution is complete when the set of clauses S is composed of:

1. A negative Horn clause
2. A set of definite Horn clauses *(which are necessarily satisfiable…)*

Complexity at step k: $|S|$ *instead* $(|S| + k)^2 * |C_i| * |C_j|$

Completeness of the SLD refutation